Quantum Coordination in Multi-User Networks

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Outline

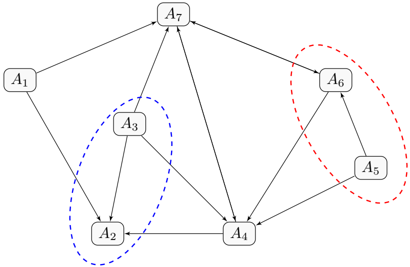
Quantum Coordination

Task Description

- Consider m users in a multi-user network
- User $i \in \{1, \ldots, m\}$ has access to a quantum system $A_i^{\otimes n}$
- Each user performs an encoding operation $\mathcal{E}_i^{\otimes n}$
- The goal is to simulate a certain quantum correlation described by $\omega_{A_1 A_2 \ldots A_m}^{\otimes n}$

Resources

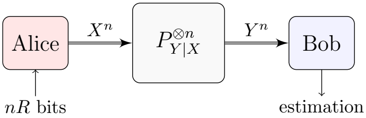
- Limited communication rates between nodes i and j; classical $R_{i,j}$ or quantum $Q_{i,j}$
- May share random bits or entangled pairs at limited rates R_0 and $E_{i,j}$ respectively



Background: Classical 2-Node Coordination

- Shannon 1948 - **Noisy Channel Coding Theorem** [1]:

$$C(\mathcal{E}) = \max_{p_X} I(X;Y),\; R \leq I(X;Y)$$



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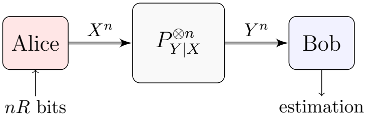
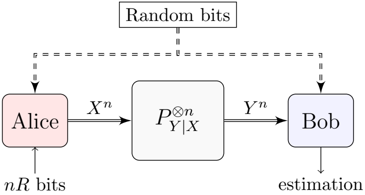


- Common randomness does not increase the capacity



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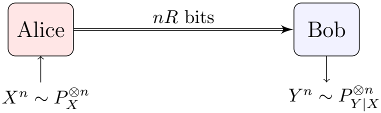
Background: Classical 2-Node Coordination (Cont.)

- Classical coordination $\iff$ Channel simulation: $P_{X^nY^n} \approx P_{XY}^{\otimes n}$



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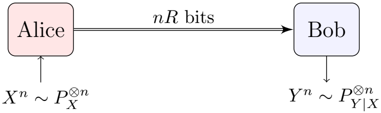
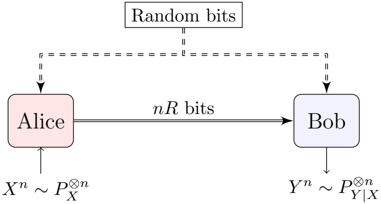


- Bennett et al., 2001 - **Classical Reverse Shannon Theorem** [2]:

$$R \geq C(\mathcal{E})$$



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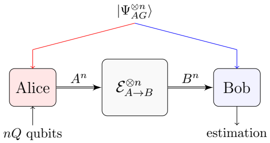
Background: Quantum 2-Node Coordination

- **Entanglement assisted quantum capacity**[2]

$$C_E(\mathcal{E}) = \frac{1}{2} \max_{\Psi_{AG}} I(G;B)_\rho, \ \rho_{BG} = \mathcal{E}_{A\to B}(\Psi_{AG})$$



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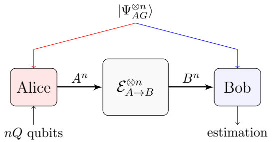
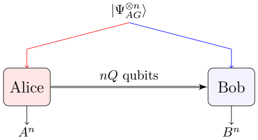


- Bennett et al. 2009 - **Quantum Reverse Shannon Theorem** [3]:

$$Q \geq C_E(\mathcal{E})$$



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Special Cases

Quantum coordination includes many models and tasks as special cases:

- Source Coding [Schumacher 1995]
- Entanglement Dilution [Hayden and Winter, 2003]
- State Merging [Horodecki 2007]
- State Redistribution [Yard and Devetak, 2009]
- Source / Channel Simulation
  - ▷ 2-node [Berta et al., 2013] [Wilde, 2018] [George and Chitambar, 2024]
  - ▷ multiple access [Nema et al., 2024]
  - ▷ non-interactive [Salehi et al., 2024]
- . . .

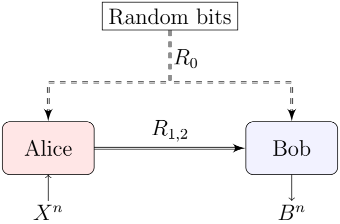
Two types of networks

We consider two types of quantum coordination networks:

1. Classical links: Generating separable correlations
2. Quantum links: Generating entanglement

Part 1: Classical Links

Two-Node Network (classical links)



- The desired state is a classical-quantum (CQ) state $\omega_{XB}^{\otimes n}$
- Common randomness (CR) is available at a rate R_0
- Alice can send classical bits to Bob at a limited rate $R_{1,2}$

Two-Node Network: Main Result

Theorem 1: Two-node coordination capacity region

The coordination capacity region for the two-node network is the set

$$\mathcal{R}_{\text{2-node}}(\omega) = \bigcup_{\mathscr{S}_{\text{2-node}}(\omega)} \left\{ \begin{array}{lrl} (R_0, R_{1,2}) : & R_{1,2} & \geq I(X;U)_\sigma \,, \\ & R_0 + R_{1,2} & \geq I(XB;U)_\sigma \end{array} \right\},$$

where $\mathscr{S}_{\text{2-node}}(\omega)$ is the set of all c-q states

$$\sigma_{XUB} = \sum_{(x,u)\in\mathcal{X}\times\mathcal{U}} p_{X,U}(x,u)\, |x\rangle\langle x|_X \otimes |u\rangle\langle u|_U \otimes \theta_B^u$$

such that

$$\sigma_{XB} = \omega_{XB}$$

for

$$|\mathcal{U}| \leq |\mathcal{X}|^2 [\dim(\mathcal{H}_B)]^2 + 1\,.$$

Two-Node Network: Corollaries

Reminder - Wyner's Common Information [1]:

$$C(X;Y) = \min_{X - U - Y} I(XY;U)$$

Corollary 1: No CR [George et al., 2023]

The coordination capacity without CR is

$$\mathsf{R}^{(0)}_{\textbf{2-node}}(\omega) = \min_{\sigma_{XUB} \in \mathscr{S}_{\textbf{2-node}}(\omega)} I(XB;U)_\sigma \,.$$

Corollary 2: Unlimited CR

The CR-assisted coordination capacity, i.e., with unlimited common randomness, is given by

$$\mathsf{R}^{(\infty)}_{\textbf{2-node}}(\omega) \triangleq \min_{\sigma_{XUB} \in \mathscr{S}_{\textbf{2-node}}(\omega)} I(X;U)_\sigma$$

[1] A. Wyner, "The common information of two dependent random variables", 1975

Two-Node Network: Analysis

Achievability proof outline:

- Select a random codebook $\mathscr{C} = \{u^n(m_0, m_{1,2})\}$ by drawing $2^{n(R_0+R_{1,2})}$ i.i.d. sequences according to $p_U^{\otimes n}(u^n) = \prod_{k=1}^n p_U(u_k)$. Let $X^n \sim p_{X|U}^{\otimes n}(x^n|u^n(m_0, m_{1,2}))$
  Given uniform $(M_0, M_{1,2})$ this induces a joint distribution $\widetilde{P}_{M_0 M_{1,2} X^n}$

- By classical resolvability, $R_{1,2} \geq I(X;U)$ guarantees

$$\mathbb{E}_{\mathscr{C}} \left\| \widetilde{P}_{M_0 X^n} - p_{M_0} \times p_X^n \right\|_1 \leq \delta$$

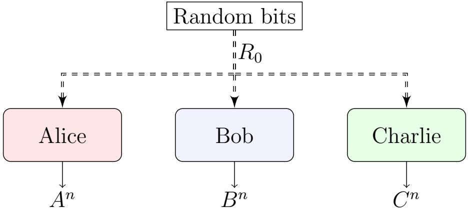
- Alice's encoder sends $m_{1,2} \sim \widetilde{P}_{M_{1,2}|X^n M_0}$
- By quantum resolvability, for $R_0 + R_{1,2} \geq I(XB;U)_\sigma$,

$$\mathbb{E}_{\mathscr{C}} \left\| \omega_{XB}^{\otimes n} - \widetilde{\tau}_{X^n B^n} \right\|_1 \leq \delta \,,$$

  where $\widetilde{\tau}$ corresponds to $X^n \sim \widetilde{P}_{X^n|M_0}$

- Using the triangle inequality, the two bounds yield distance $\leq 2\delta$

No-Communication Network (classical links)



- The desired state is a separable quantum state $\omega_{ABC}^{\otimes n}$
- Common randomness (CR) is available to all users at a rate R_0
- No communication is allowed between the users

⋆ independently considered by [George and Chitambar, 2024]

No-Communication Network: Main result

Theorem 2: No-communication coordination capacity

The coordination capacity for the no-communication network is the set

$$\mathsf{C}_{\mathsf{NC}}(\omega) = \inf_{\sigma_{UABC} \in \mathscr{S}_{\mathbf{NC}}(\omega)} I(U;ABC)_\sigma \,,$$

where $\mathscr{S}_{\mathsf{NC}}(\omega)$ is the set of all c-q states

$$\sigma_{UABC} = \sum_{u \in \mathcal{U}} p_U(u) \; |u\rangle\langle u|_U \otimes \theta_A^u \otimes \theta_B^u \otimes \theta_C^u \,,$$

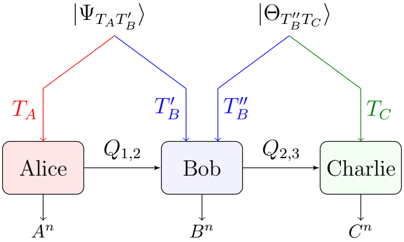
such that

$$\sigma_{ABC} = \omega_{ABC} \,.$$

- **Remark:** Since the CR is classical, it cannot be used to create entanglement. We can only simulate separable states.

Part 2: Quantum Links

Cascade Network (quantum links)



- The desired state is $\omega_{ABC}^{\otimes n}$
- Alice and Bob share entangled pairs at a rate $E_{1,2}$, Bob and Charlie share entangled pairs at a rate $E_{2,3}$
- Alice can send qbits to Bob at a limited rate $Q_{1,2}$, and Bob can send qubits to Charlie at a rate $Q_{2,3}$

Cascade Network: Main Result

Theorem 3: Cascade coordination capacity region

Let $|\omega_{RABC}\rangle$ be a purification of ω_{ABC}. The coordination capacity region for the cascade network is given by the set

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ (Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3}) : \begin{array}{rl} Q_{1,2} & \geq \frac{1}{2} I(BC;R)_\omega \,, \\ Q_{1,2} + E_{1,2} & \geq H(BC)_\omega \,, \\ Q_{2,3} & \geq \frac{1}{2} I(C;RA)_\omega \,, \\ Q_{2,3} + E_{2,3} & \geq H(C)_\omega \end{array} \right\} .$$

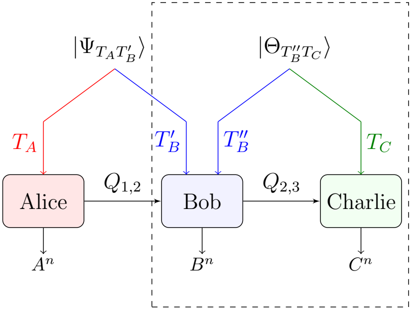
- Achievability relies on state redistribution [Yard and Devatak, 2009] [4] (decoupling).
- Converse follows from entanglement-assisted capacity results.

[4]. T. Yard and I. Devetak, "Optimal quantum source coding with quantum side information at the encoder and decoder", 2009

Cascade Network: Analysis

Converse proof outline

- Consider Alice's communication and entanglement rates, $Q_{1,2}$ and $E_{1,2}$.
- We may view the entire encoding operation of Bob and Charlie as a black box whose input and output are $(M_{1,2}, T'_B)$ and (B^n, C^n), respectively.



Cascade Network: Analysis (Cont.)

- By the definition of $(Q_{1,2}, E_{1,2})$

$$
\begin{aligned}
2n(Q_{1,2}+E_{1,2}) &= 2\left[\log\dim(\mathcal{H}_{M_{\mathbf{1},\mathbf{2}}})+\log\dim(\mathcal{H}_{T'_B})\right] \\
&\geq I(M_{1,2}T'_B;A^nR^n)_\rho \\
&\overset{(a)}{\geq} I(B^nC^n;A^nR^n)_\rho \\
&\overset{(b)}{\geq} I(B^nC^n;A^nR^n)_{\omega^{\otimes n}}-n\alpha_n \\
&= n[I(BC;AR)_\omega-\alpha_n] \\
&\overset{(c)}{=} 2n\left(H(BC)_\omega-\frac{1}{2}\alpha_n\right) .
\end{aligned}
$$

by

(a) data processing inequality
(b) Alicki-Fannes-Winter inequality (entropy continuity)
(c) $|\omega_{RABC}\rangle$ is pure

Cascade Network: Analysis (Cont.)

- Based on the entanglement assisted capacity result,

$$\begin{aligned} Q_{1,2}\,[q \to q]_{A\to BC} &\geq \langle \omega_{RBC} \rangle \\ &\equiv \langle \mathrm{tr}_A : \omega_{RABC} \rangle \\ &\geq \frac{1}{2} I(BC;R)_\omega \, [q \to q]_{A\to BC} \end{aligned}$$

- Similar arguments hold for the rate pair $(Q_{2,3}, E_{2,3})$.

Cascade: Corollaries

Corollary 3: Simulating a pure state

For a pure state $|\omega_{ABC}\rangle$, The coordination capacity region for the cascade network is given by the set

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{ll} (Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3}) : & \begin{array}{rl} Q_{1,2} + E_{1,2} & \geq H(BC)_\omega \, , \\ Q_{2,3} & \geq \frac{1}{2} I(C;A)_\omega \, , \\ Q_{2,3} + E_{2,3} & \geq H(C)_\omega \end{array} \end{array} \right\} .$$

Cascade example 1

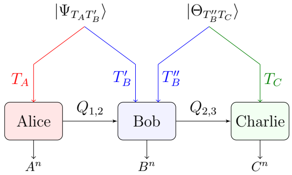
Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$ be Hilbert spaces of dimension 3, i.e., qutrits. Consider the simulation of a mixed state,

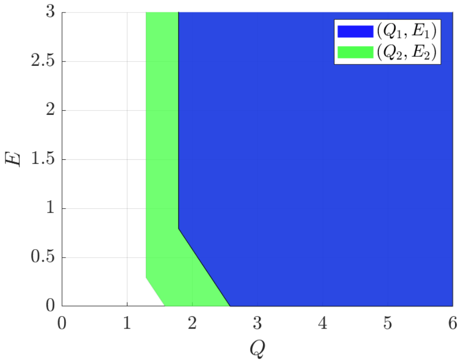
$$\omega_{ABC} = \frac{1}{6}\left(|123\rangle\langle 123| + |132\rangle\langle 132| + |213\rangle\langle 213| + |231\rangle\langle 231| + |312\rangle\langle 312| + |321\rangle\langle 321|\right)$$

According to our results, $\omega_{ABC}^{\otimes n}$ can be simulated if and only if the rate tuple $(Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3})$ belongs to the following set:

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{ll} (Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3}) : & \begin{array}{rl} Q_{1,2} & \geq 1.7925, \\ Q_{1,2} + E_{1,2} & \geq 2.5850\,, \\ Q_{2,3} & \geq 1.2925\,, \\ Q_{2,3} + E_{2,3} & \geq 1.5850\,. \end{array} \end{array} \right\} .$$

Cascade example 1 (Cont.)





Cascade example 2

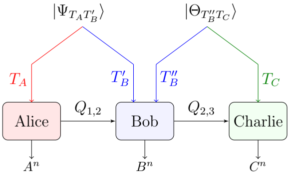
Now we consider the simulation of a pure state,

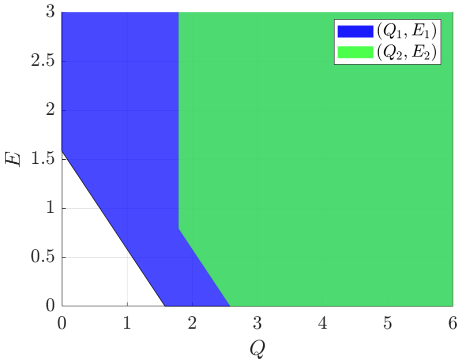
$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{6}}\left(|012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle\right)$$

According to the corollary, $|\psi_{ABC}\rangle^{\otimes n}$ can be simulated if and only if the rate tuple $(Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3})$ belong to the following set,

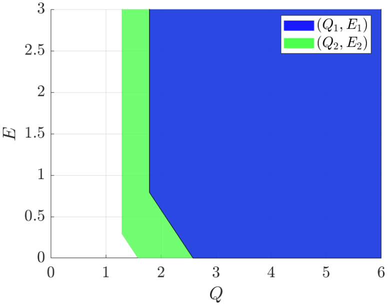
$$\mathcal{Q}_{\text{Cascade}}(\psi) = \left\{ \begin{array}{lrl} (Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3}) : & Q_{1,2} + E_{1,2} & \geq 1.5850\,, \\ & Q_{2,3} & \geq 0.7925\,, \\ & Q_{2,3} + E_{2,3} & \geq 1.5850\,. \end{array} \right\}.$$

Cascade example 2 (Cont.)

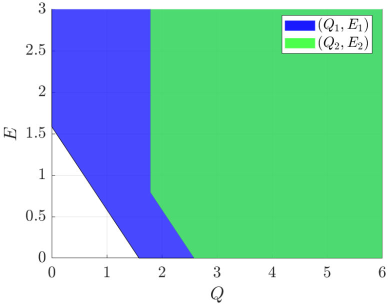




Cascade examples (Cont.)

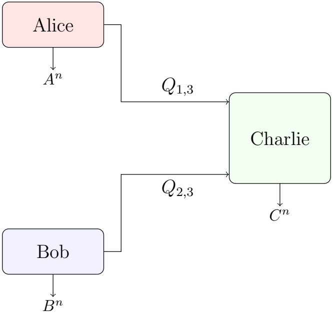


Mixed State



Pure State

Multiple-Access Network (quantum links)



- The desired state is $|\omega_{ABC}\rangle^{\otimes n}$
- Alice and Bob can send qubits to Charlie rates $Q_{1,3}$ and $Q_{2,3}$ respectively
- No common randomness nor entanglement assistance are allowed

Multiple-Access Network: Main Result

Theorem 4: Multiple-access coordination capacity region

The coordination capacity region for the multiple access network is given by the set

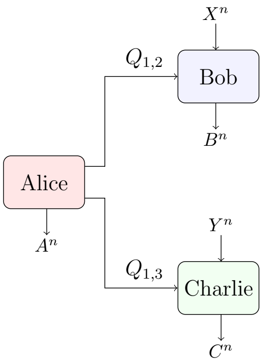
$$\mathcal{Q}_{\text{MAC}}(\omega) = \left\{ \begin{array}{ll} (Q_{1,3}, Q_{2,3}) \in \mathbb{R}^2 : & \begin{array}{ll} Q_{1,3} & \geq H(A)_\omega \,, \\ Q_{2,3} & \geq H(B)_\omega \end{array} \end{array} \right\} .$$

Remark 1: Schumacher's compression protocol is optimal in this network.

Remark 2: Since there is no cooperation between the transmitters Alice and Bob, we can only simulate states $|\omega_{ABC}\rangle$ for which there exists an isometry $V_{C\to C_1 C_2}$ such that

$$(\mathbb{I} \otimes V_{C\to C_1 C_2}) |\omega_{ABC}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle$$

Broadcast Network (quantum links)



- The desired state is $\omega_{XYABC}^{\otimes n}$
- Bob and Charlie have access to a classical information sequences X^n and Y^n respectively.
- Alice can send qubits to Bob and Charlie at rate $Q_{1,2}$ and $Q_{1,3}$ respectively.

Broadcast Network: Main Result

Theorem 5: Broadcast coordination capacity region

The coordination capacity region for the broadcast network is given by the set

$$\mathcal{Q}_{\mathsf{BC}}(\omega) = \left\{ \begin{array}{lll} (Q_{1,2}, Q_{1,3}) \in \mathbb{R}^2 : & Q_{1,2} & \geq H(B|X)_\omega \,, \\ & Q_{1,3} & \geq H(C|Y)_\omega \end{array} \right\} .$$

Remarks:

- The problem is closely related to quantum source coding with classical side information [Khanian and Winter, 2020], except that Alice does not know X^n, Y^n.
- Since Alice has no access to X^n nor Y^n, coordination can only be achieved for states ω_{XYABC} such that

$$\omega_{XYA} = \omega_{XY} \otimes \omega_A \,.$$

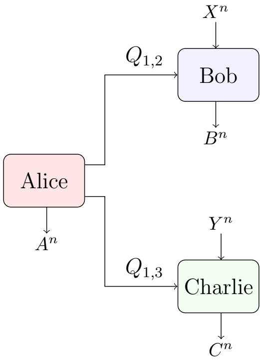
Broadcast Network & Nonlocal Games

- Quantum correlations can increase the winning probability in various nonlocal games.

Broadcast Network & Nonlocal Games

- Quantum correlations can increase the winning probability in various nonlocal games.
- For example, the CHSH game and the Magic Square game.

Broadcast Network & Nonlocal Games (Cont.)



- **Single shot game ($n = 1$):** The broadcast network model can represent a nonlocal game, where a referee provides random queries X to Bob and Y to Charlie. Bob and Charlie respond with classical B and C, winning if (X, Y, B, C) satisfies a specific condition (e.g., in the CHSH game, $X \wedge Y = B \oplus C$).

Broadcast Network & Nonlocal Games (Cont.)

- **Sequential game:** In the sequential version, the players repeat the game n times Let $\mathscr{S}(\gamma)$ denote the set of correlations $P_{BC|XY}$ that win the game with probability γ.
  Based on our results, the game can be won with probability γ if and only if Alice can send qubits to Bob and Charlie at rates $Q_{1,2}$ and $Q_{1,3}$ that satisfy the constraints

$$Q_{1,2} \geq H(B|X)_\omega \,,\; Q_{1,3} \geq H(C|Y)_\omega \,.$$

with respect to some correlation $P_{BC|XY} \in \mathscr{S}(\gamma)$.

Broadcast Network: Analysis

Proof rough idea:

- Consider a classical-quantum state,

$$\omega_{XYABC} = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{XY}(x,y) \, |x,y\rangle\langle x,y|_{X,Y} \otimes \left|\sigma_{ABC}^{(x,y)}\right\rangle\left\langle\sigma_{ABC}^{(x,y)}\right| \,,$$

- Consider a spectral decomposition of the reduced state of Bob,

$$\sigma_B^{(x)} = \sum_{z \in \mathcal{Z}} p_{Z|X}(z|x) \, |\psi_{x,z}\rangle\langle\psi_{x,z}|$$

- For every sequence $z^n \in \mathcal{Z}^n$, assign a bin index $m_1(z^n)$ from $[2^{nQ_1}]$, uniformly at random.

- Use Schumacher's compression algorithm with respect to the binning function.

1. Background
2. Networks and Results
3. Summary and Discussion

Summary and Discussion

- We consider quantum coordination in two types of quantum networks: networks with classical links, and others with quantum links.
- The network topology and the allowed resources dictate the types of correlations that can be simulated.
- Tradeoff between entanglement assistance and communication requirements in the cascade network.
- Relation to quantum nonlocal games.
- Most network analysis can be generalized to a general number of users.

Future Work

- One shot coordination
- Cascade network with classical links
- Implications on conference key distribution
- Bosonic states (continuous variables)

Full Papers

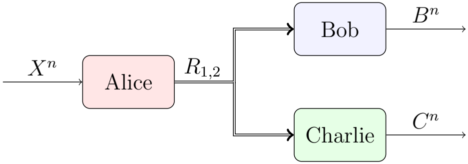
1. H. Nator and U. Pereg, "Coordination capacity for classical-quantum correlations," arXiv preprint arXiv: 2404.18297, 2024.

2. H. Nator and U. Pereg, "Entanglement coordination rates in multi-user networks", arXiv preprint arXiv: 2403.11893, 2024.

Thank You

Additional Results and Analysis

Broadcast Network (classical links)



- The desired state is a classical-quantum-quantum state $\omega_{XBC}^{\otimes n}$
- Common randomness (CR) is available to all users at a rate R_0
- Alice can send classical bits to Bob and Charlie at a rate $R_{1,2}$

Broadcast Network: Main Result

Theorem 1: Broadcast Network coordination capacity

The coordination capacity for the broadcast network is the set

$$\mathcal{R}_{\text{BC}}(\omega) = \bigcup_{\mathscr{S}_{\mathbf{BC}}(\omega)} \left\{ \begin{array}{ll} (R_0, R_{1,2}) : & \begin{array}{rl} R_1 & \geq I(X;U)_\sigma \,, \\ R_0 + R_{1,2} & \geq I(XBC;U)_\sigma \end{array} \end{array} \right\},$$

where $\mathscr{S}_{\text{2-BC}}(\omega)$ be the set of all c-q states

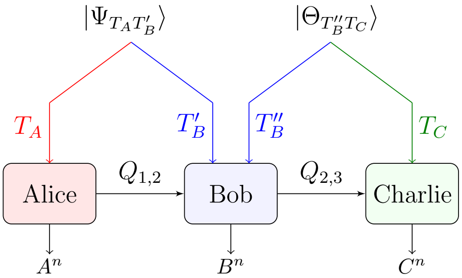
$$\sigma_{XUBC} = \sum_{\substack{(x,u)\in \\ \mathcal{X}\times\mathcal{U}}} p_{XU}(x,u)\, |x\rangle\langle x|_X \otimes |u\rangle\langle u|_U \otimes \theta_B^u \otimes \eta_C^u$$

such that

$$\sigma_{XBC} = \omega_{XBC}\,.$$

- **Remark:** Alice's encoding is classical, hence, she cannot distribute entanglement.

Cascade Network (quantum links)



Theorem 2: Cascade coordination capacity region

Let $|\omega_{RABC}\rangle$ be a purification of ω_{ABC}. The coordination capacity region for the cascade network is given by the set

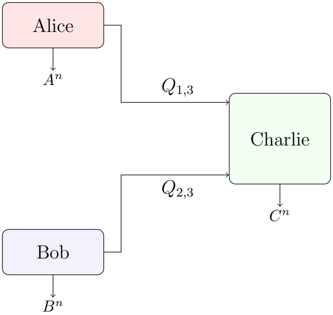
$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{lrl} (Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3}) : & Q_{1,2} & \geq \frac{1}{2} I(BC;R)_\omega \,, \\ & Q_{1,2} + E_{1,2} & \geq H(BC)_\omega \,, \\ & Q_{2,3} & \geq \frac{1}{2} I(C;RA)_\omega \,, \\ & Q_{2,3} + E_{2,3} & \geq H(C)_\omega \end{array} \right\} .$$

- **Remark:** The optimal cost of quantum communication and entanglement assistance are [4]:

$$Q^* = \frac{1}{2} I(B;GR), \quad E^* = \frac{1}{2} I(A;B) - \frac{1}{2} I(G;B) \,.$$

[4]. T. Yard and I. Devetak, "Optimal quantum source coding with quantum side information at the encoder and decoder", 2009

Multiple-Access Network (quantum links)



Multiple-Access Network (quantum links)

In the multiple-access network, Alice sends nQ_1 qubits to Charlie, while Bob sends nQ_2 qubits to Charlie. Specifically, Alice and Bob apply the encoding maps, preparing $\rho^{(1)}_{A^n M_1} \otimes \rho^{(2)}_{B^n M_2}$, where

$$\rho^{(1)}_{A^n M_1} = \mathcal{E}_{A^n \to A^n M_1}(\omega_A^{\otimes n}) \,,\; \rho^{(2)}_{B^n M_2} = \mathcal{F}_{B^n \to B^n M_2}(\omega_B^{\otimes n}) \,.$$

As Charlie receives M_1 and M_2, he applies his encoding map, which yields the final state,

$$\widehat{\rho}_{A^n B^n C^n} = (\mathrm{id}_{A^n B^n} \otimes \mathcal{D}_{M_1 M_2 \to C^n})\,(\rho^{(1)}_{A^n M_1} \otimes \rho^{(2)}_{B^n M_2}) \,.$$

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The ultimate goal of the coordination protocol is that the final state of $\widehat{\rho}_{A^n B^n C^n}$, is arbitrarily close to the desired state $\omega_{ABC}^{\otimes n}$.

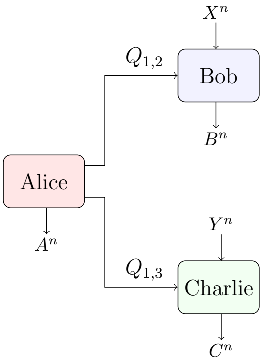
Multiple-Access Network (quantum links)

Remark Notice that since Charlie only acts on M_1 and M_2 which are encoded separately without coordination, we have $\widehat{\rho}_{A^nB^n} = \rho^{(1)}_{A^n} \otimes \rho^{(2)}_{B^n}$. Therefore, it is only possible to simulate states ω_{ABC} such that $\omega_{AB} = \omega_A \otimes \omega_B$. Since all purifications are isometrically equivalent [1, Theorem 5.1.1] there exists an isometry $V_{C\to C_1C_2}$ such that

$$(\mathbb{I} \otimes V_{C\to C_1C_2}) |\omega_{ABC}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle \tag{3}$$

where $|\phi_{AC_1}\rangle$ and $|\chi_{BC_2}\rangle$ are purifications of ω_A and ω_B, respectively. If ω_{ABC} cannot be decomposed using an isometry, then coordination is impossible in the multiple-access network.

Broadcast Network (quantum links)



Achievability Proof for the Broadcast Network - Prerequisites

We show achievability using a quantum version of the binning technique.

- Consider a classical-quantum state,

$$\omega_{XYABC} = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p_{XY}(x,y) \, |x,y\rangle\langle x,y|_{X,Y} \otimes \left|\sigma_{ABC}^{(x,y)}\right\rangle\left\langle\sigma_{ABC}^{(x,y)}\right| \, ,$$

corresponding to an ensemble of states $\left\{p_{XY}, \left|\sigma_{ABC}^{(x,y)}\right\rangle\right\}$ in $\Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$.

- Let $\varepsilon_i, \delta > 0$ be arbitrarily small. Define the average states,

$$\sigma_{AB}^{(x)} = \sum_{y\in\mathcal{Y}} p_{Y|X}(y|x)\sigma_{AB}^{(x,y)} \, ,$$
$$\sigma_{AC}^{(y)} = \sum_{x\in\mathcal{X}} p_{X|Y}(x|y)\sigma_{AC}^{(x,y)} \, .$$

Achievability Proof for the Broadcast Network - Prerequisites

- Consider a spectral decomposition of the reduced states of Bob and Charlie,

$$\sigma_B^{(x)} = \sum_{z\in\mathcal{Z}} p_{Z|X}(z|x)\,|\psi_{x,z}\rangle\langle\psi_{x,z}|\ ,$$
$$\sigma_C^{(y)} = \sum_{w\in\mathcal{W}} p_{W|Y}(w|y)\,|\phi_{y,w}\rangle\langle\phi_{y,w}|\ .$$

- $\{|\psi_{x,z}\rangle\}_z, \{|\phi_{y,w}\rangle\}_w$ are orthonormal bases for $\mathcal{H}_B$, $\mathcal{H}_C$, respectively, for every given $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

- We can also assume that the different bases are orthogonal to each other by requiring that Bob and Charlie encode on a different Hilbert space for every value of (x, y).

- **Notation:**
  - ▷ $T_\delta^{X^n}$ denotes the δ-typical set with respect to p_X, and $T_\delta^{Z^n|x^n}$ is the conditional δ-typical set with respect to p_{XZ}, given $x^n \in T_\delta^{X^n}$.
  - ▷ $\Delta(\mathcal{H})$ is the set of density operators in $\mathcal{H}$.

Achievability Proof for the Broadcast Network - Protocol

Classical Codebook Generation:

- For every sequence $z^n \in \mathcal{Z}^n$, assign an index $m_1(z^n)$, uniformly at random from $[2^{nQ_1}]$.
- A bin $\mathfrak{B}_1(m_1)$ is defined as the subset of sequences in $\mathcal{Z}^n$ that are assigned the same index m_1, for $m_1 \in [2^{nQ_1}]$.
- The codebook is revealed to all parties.

Achievability Proof for the Broadcast Network - Protocol

Encoding:

- **Alice**
  - ▷ prepares $\omega_{A\bar{B}\bar{C}}^{\otimes n}$ locally, where $\bar{B}^n\bar{C}^n$ are her ancillas, without any correlation with X^n and Y^n.
  - ▷ She applies the encoding channel $\mathcal{E}^{(1)}_{\bar{B}^n\to M_1} \otimes \mathcal{E}^{(2)}_{\bar{C}^n\to M_2}$,

$$\mathcal{E}^{(1)}_{\bar{B}^n\to M_1}(\rho_1) = \sum_{x^n\in\mathcal{X}^n} p_X^{\otimes n}(x^n) \sum_{z^n\in\mathcal{Z}^n} \langle\psi_{x^n,z^n}|\,\rho_1\,|\psi_{x^n,z^n}\rangle\, |m_1(z^n)\rangle\langle m_1(z^n)| \ ,$$

$$\mathcal{E}^{(2)}_{\bar{C}^n\to M_2}(\rho_2) = \sum_{y^n\in\mathcal{Y}^n} p_Y^{\otimes n}(y^n) \sum_{w^n\in\mathcal{W}^n} \langle\phi_{y^n,w^n}|\,\rho_2\,|\phi_{y^n,w^n}\rangle\, |m_2(w^n)\rangle\langle m_2(w^n)| \ ,$$

for $\rho_1 \in \Delta(\mathcal{H}_B^{\otimes n})$, $\rho_2 \in \Delta(\mathcal{H}_C^{\otimes n})$, and transmits M_1 and M_2 to Bob and Charlie, respectively.

Achievability Proof for the Broadcast Network - Protocol

- **Bob** applies the following encoding channel,

$$\mathcal{F}^{(x^n)}_{M_1 \to B^n}(\rho_{M_1}) = \sum_{m_1=1}^{2^{nQ_1}} \langle m_1 | \rho_{M_1} | m_1 \rangle \left(\frac{1}{\left| T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1) \right|} \sum_{z^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1)} |\psi_{x^n,z^n}\rangle\langle\psi_{x^n,z^n}| \right)$$

- **Charlie** encodes in a similar manner.

Error Analysis

- Due to the code construction, it suffices to consider the individual errors of Bob and Charlie,

$$\frac{1}{2}\left\|\omega_{XAB}^{\otimes n} - \left(\mathcal{F}_{X^nM_1\to X^nB^n} \circ \mathcal{E}^{(1)}_{\bar{B}^n\to M_1}\right)(\omega_X^{\otimes n} \otimes \omega_{A\bar{B}}^{\otimes n})\right\|_1 ,$$
$$\frac{1}{2}\left\|\omega_{YAC}^{\otimes n} - \left(\mathcal{D}_{Y^nM_2\to Y^nC^n} \circ \mathcal{E}^{(2)}_{\bar{C}^n\to M_2}\right)(\omega_Y^{\otimes n} \otimes \omega_{A\bar{C}}^{\otimes n})\right\|_1 ,$$

  respectively, where we use the short notation $\mathcal{E}^{(1)}_{\bar{B}^n\to M_1} \equiv \mathrm{id}_{X^nA^n} \otimes \mathcal{E}^{(1)}_{\bar{B}^n\to M_1}$, and similarly for the other encoding maps.
- We now focus on Bob's error.

- Consider a given codebook $\mathscr{C}_1 = \{m_1(z^n)\}$. Alice encodes M_1 by

$$\mathcal{E}^{(1)}_{\bar{B}^n \to M_1}(\omega_{AB}^{\otimes n}) = \sum_{\tilde{x}^n \in \mathcal{X}^n} p_X^{\otimes n}(\tilde{x}^n) \sum_{z^n \in \mathcal{Z}^n} \langle \psi_{\tilde{x}^n, z^n} | \, \omega_{AB}^{\otimes n} \, | \psi_{\tilde{x}^n, z^n} \rangle \, |m_1(z^n)\rangle\langle m_1(z^n)| \, .$$

- By the weak law of large numbers, this state is ε_1-close in trace distance to

$$\begin{aligned}\rho^{(1)}_{A^n M_1} &= \sum_{\tilde{x}^n \in T_\delta^{X^n}} p_X^{\otimes n}(\tilde{x}^n) \sum_{z^n \in T_\delta^{Z^n | \tilde{x}^n}} \langle \psi_{\tilde{x}^n, z^n} | \, \sigma^{(\tilde{x}^n)}_{A^n \bar{B}^n} \, | \psi_{\tilde{x}^n, z^n} \rangle \, |m_1(z^n)\rangle\langle m_1(z^n)| \\ &= \sum_{x^n \in T_\delta^{X^n}} p_X^{\otimes n}(x^n) \rho^{(1|x^n)}_{A^n M_1} \, ,\end{aligned}$$

for sufficiently large n, where we have defined

$$\rho^{(1|x^n)}_{A^n M_1} = \sum_{z^n \in T_\delta^{Z^n | x^n}} \langle \psi_{x^n, z^n} | \, \sigma^{(x^n)}_{A^n \bar{B}^n} \, | \psi_{x^n, z^n} \rangle \, |m_1(z^n)\rangle\langle m_1(z^n)| \, .$$

Achievability Proof for the Broadcast Network - Error Analysis

- Let $x^n \in T_\delta^{X^n}$. After Bob encodes B^n, we have

$$\mathcal{F}^{(x^n)}_{M_1 \to B^n}\left(\rho^{(1|x^n)}_{A^n M_1}\right) = \sum_{z^n \in T_\delta^{Z^n|x^n}} \langle \psi_{x^n,z^n} | \, \sigma^{(x^n)}_{A^n \bar{B}^n} \, | \psi_{x^n,z^n} \rangle \, \mathcal{F}^{(x^n)}_{M_1 \to B^n}(|m_1(z^n)\rangle\langle m_1(z^n)|) \, .$$

- By the definition of Bob's encoding channel, $\mathcal{F}^{(x^n)}_{M_1 \to B^n}$,

$$\mathcal{F}^{(x^n)}_{M_1 \to B^n}(|m_1(z^n)\rangle\langle m_1(z^n)|) = \frac{1}{\left| T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n)) \right|} \sum_{\tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))} |\psi_{x^n,\tilde{z}}\rangle\langle \psi_{x^n,\tilde{z}}| \, .$$

Achievability Proof for the Broadcast Network - Error Analysis

- Substituting in

$$\mathcal{F}^{(x^n)}_{M_{\mathbf{1}}\to B^n}\left(|m_1(z^n)\rangle\langle m_1(z^n)|\right) = \frac{1}{\left|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))\right|} \sum_{\tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_{\mathbf{1}}(m_{\mathbf{1}}(z^n))} |\psi_{x^n,\tilde{z}}\rangle\langle\psi_{x^n,\tilde{z}}| \ ,$$

- we have

$$\mathcal{F}^{(x^n)}_{M_{\mathbf{1}}\to B^n}\left(\rho^{(\mathbf{1}|x^n)}_{A^n M_{\mathbf{1}}}\right) = \sum_{z^n \in T_\delta^{Z^n|x^n}} \langle\psi_{x^n,z^n}|\, \sigma^{(x^n)}_{A^n\bar{B}^n}\, |\psi_{x^n,z^n}\rangle \otimes \left[\frac{1}{\left|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))\right|} \sum_{\tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_{\mathbf{1}}(m_{\mathbf{1}}(z^n))} |\psi_{x^n,\tilde{z}^n}\rangle\langle\psi_{x^n,\tilde{z}^n}|\right].$$

Achievability Proof for the Broadcast Network - Error Analysis

- Based on the classical result [5], the random codebook $\mathscr{C}_1$ satisfies that

$$\Pr_{\mathscr{C}_1}\left(\exists \tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n)) : \tilde{z}^n \neq z^n\right) \leq \varepsilon_2$$

given $z^n \in T_\delta^{Z^n|x^n}$, for sufficiently large n, provided that the codebook size is at least $2^{n(H(Z|X)+\varepsilon_3)}$, where $H(Z|X)$ denotes the classical conditional entropy. As $|\mathscr{C}_1| = 2^{nQ_1}$, this holds if

$$\begin{aligned} Q_1 &> H(Z|X) + \varepsilon_3 \\ &= H(B|X)_\omega + \varepsilon_3 \,. \end{aligned}$$

- Therefore the set in $T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))$, consists of the sequence z^n alone.

[5] A. E. Gamal and Y.H. Kim, "Network Information Theory", 2011.

Achievability Proof for the Broadcast Network - Error Analysis

- Therefore, the overall state

$$\mathcal{F}^{(x^n)}_{M_{\mathbf{1}}\to B^n}\left(\rho^{(\mathbf{1}|x^n)}_{A^n M_{\mathbf{1}}}\right) = \sum_{z^n \in T_\delta^{Z^n|x^n}} \langle \psi_{x^n,z^n} |\, \sigma^{(x^n)}_{A^n \bar{B}^n} \,| \psi_{x^n,z^n} \rangle \otimes \left[\frac{1}{\left| T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n)) \right|} \sum_{\tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_{\mathbf{1}}(m_{\mathbf{1}}(z^n))} |\psi_{x^n,\tilde{z}^n}\rangle\langle\psi_{x^n,\tilde{z}^n}| \right].$$

  is identical to the post-measurement state after a typical subspace measurement on B^n, with respect to the conditional δ-typical set $T_\delta^{Z^n|x^n}$.
- Based on the gentle measurement lemma, this state is ε_4-close to $\sigma^{(x^n)}_{AB}$, for sufficiently large n.

Achievability Proof for the Broadcast Network - Error Analysis

- Therefore, by the triangle inequality and total expectation formula,

$$\left\| \omega_{XAB}^{\otimes n} - \mathbb{E}_{\mathscr{C}_1} \left(\mathcal{F}_{X^n M_1 \to X^n B^n} \circ \mathcal{E}^{(1)}_{\bar{B}^n \to M_1} \right) \left(\omega_X^{\otimes n} \otimes \omega_{A\bar{B}}^{\otimes n} \right) \right\|_1$$
$$\leq \sum_{x^n \in \mathcal{X}^n} p_X^{\otimes n}(x^n) \cdot \mathbb{E}_{\mathscr{C}_1} \left\| \sigma_{A^n B^n}^{(x^n)} - \left(\mathcal{F}^{(x^n)}_{M_1 \to B^n} \circ \mathcal{E}^{(1)}_{\bar{B}^n \to M_1} \right) \left(\sigma_{A^n B^n}^{(x^n)} \right) \right\|_1$$
$$\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_4 \,.$$

- By symmetry, Charlie's error tends to zero as well, provided that $Q_2 \geq H(C|Y)_\omega + \varepsilon_5$.
- The achievability proof follows by taking $n \to \infty$ and then $\varepsilon_j, \delta \to 0$.

Converse The converse proof follows the lines of the converse proof of the state redistribution theorem [4].